Vježbe

Grupa permutacija

$X \neq \emptyset$, $|X| = n$, $X = \{1, 2, \dots, n\}$

$\pi : X \to X$ bijektivno

$$\operatorname{sgn} \pi = \prod_{i<j} \frac{\pi(j) - \pi(i)}{j - i} \in \{-1, 1\}$$

1. a) $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

b) $\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n & n-1 & n-2 & \dots & 1 \end{pmatrix}$

a) $\left.\begin{matrix} 1 & 2 & + \\ 1 & 3 & - \\ 2 & 3 & - \end{matrix}\right\} + \quad \operatorname{sgn} \pi = 1$

b) $\operatorname{sgn} \pi = (-1)^k$, k – broj inverzija

$$k = \binom{n}{2} = \frac{n(n-1)}{2}$$

$$\operatorname{sgn} \pi = (-1)^{\frac{n(n-1)}{2}}$$

T $\operatorname{sgn} : S_n \to \underbrace{\{-1, 1\}}_{C_2}$

$\operatorname{sgn}(\pi_1 \circ \pi_2) = \operatorname{sgn} \pi_1 \cdot \operatorname{sgn} \pi_2$

sgn je homomorfizam između $(S_n, \circ)$ i $(C_2, \cdot)$

2. Dokazati da je za svaku permutaciju $\alpha \in S_n$ $sgn(\alpha) = sgn(\alpha^{-1})$.

$\alpha \circ \alpha^{-1} = id_X$

$sgn(\alpha \circ \alpha^{-1}) = sgn(id_X)$

$sgn(\alpha) \cdot sgn(\alpha^{-1}) = 1$

Ako je $sgn(\alpha) = 1$ onda $sgn(\alpha^{-1}) = 1$.
Ako je $sgn(\alpha) = -1$ onda $sgn(\alpha^{-1}) = -1$.

3. a) Proizvod dvije parne (neparne) permutacije je parna permutacija.

b) Proizvod parne i neparne permutacije je neparna permutacija.

c) $\alpha, \beta \in S_n$ $\quad sgn(\alpha^{-1} \circ \beta \circ \alpha) = sgn(\beta)$

a) $\alpha, \beta \in S_n$ t.d. $sgn\alpha = sgn\beta = 1$

$sgn(\alpha \circ \beta) = sgn(\alpha) \cdot sgn(\beta) = 1 \cdot 1 = 1 \Rightarrow$

$\Rightarrow \alpha \circ \beta$ parna

slično za $sgn\alpha = sgn\beta = -1$

b) $\alpha \in S_n$ $\quad sgn(\alpha) = 1$
$\beta \in S_n$ $\quad sgn(\beta) = -1$

$\text{sgn}(\alpha \circ \beta) = \text{sgn}\,\alpha \cdot \text{sgn}\,\beta = 1 \cdot (-1) = -1 \Rightarrow \alpha \circ \beta$ непарна

c) $\alpha, \beta \in S_n$

$$\text{sgn}(\alpha^{-1} \circ \beta \circ \alpha) = \underbrace{\text{sgn}(\alpha^{-1}) \cdot \text{sgn}(\beta) \cdot \text{sgn}(\alpha)}_{1} = \text{sgn}(\beta)$$

4. $\text{sgn}: S_n \xrightarrow{\text{hom.}} C_2 = \{-1, 1\}$.

$$S_n / \text{Ker}(\text{sgn}) \cong \text{Im}(\text{sgn})$$

$$\text{Ker}(\text{sgn}) = \{\pi \in S_n \mid \text{sgn}(\pi) = 1\} = A_n$$

$\text{Im}(\text{sgn}) = C_2$ за $n > 1$

$$\boxed{S_n / A_n \cong C_2} \qquad A_n \trianglelefteq S_n$$

$$\frac{\overbrace{|S_n|}^{= n!}}{|A_n|} = 2 \Rightarrow |A_n| = \frac{n!}{2}$$

Dokazati da je $A_n \trianglelefteq S_n$.

? $A_n \leq S_n$?

$\alpha, \beta \in A_n \quad \text{sgn}(\alpha) = \text{sgn}(\beta) = 1$

? $\alpha \circ \beta^{-1} \in A_n$?

$$\text{sgn}(\alpha \circ \beta^{-1}) = \text{sgn}(\alpha) \cdot \text{sgn}(\beta^{-1}) = \text{sgn}(\alpha) \cdot \text{sgn}(\beta) = 1$$

$\Rightarrow \alpha \circ \beta^{-1} \in A_n$

$\alpha \in S_n, \beta \in A_n$

$? \ \alpha^{-1} \circ \beta \circ \alpha \in A_n \ ?$

$sgn(\alpha^{-1} \circ \beta \circ \alpha) = sgn(\beta) = 1 \Rightarrow \alpha^{-1} \circ \beta \circ \alpha \in A_n$

5. Neka je $\alpha \in S_n$. Dokazati da je $\underbrace{\alpha \circ \dots \circ \alpha}_{n!} = id_X$.

$(S_n, \circ)$ - grupa

$|S_n| = n!$

$ord(\alpha) = |\langle \alpha \rangle| \,\Big|\, |S_n|$

$ord(\alpha) \mid n!$ (Lagranžova teorema)

$m = ord(\alpha) \Rightarrow n! = m \cdot q$

$$\underbrace{\alpha \circ \dots \circ \alpha}_{n!} = \alpha^{n!} = \alpha^{m \cdot q} = (\alpha^m)^q = (id_X)^q = id_X$$

6. Predstaviti α kao proizvod transpozicija, a zatim naći znak.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

$$\alpha = (1 \ 4 \ 2 \ 3) \stackrel{?}{=} (1 \ 4) \circ (4 \ 2) \circ (2 \ 3)$$

$\alpha(1) = 4$
$\alpha(2) = 3$
$\alpha(3) = 1$
$\alpha(4) = 2$

$\alpha = \pi_1 \circ \pi_2 \circ \pi_3$

$sgn(\alpha) = sgn(\pi_1) \cdot sgn(\pi_2) \cdot sgn(\pi_3) = -1$

1	2	−
1	3	−
1	4	−
2	3	−
2	4	−
3	4	+

7. a) $\pi = (a_1 a_2 \cdots a_n) = (a_1 a_n) \circ (a_1 a_{n-1}) \circ (a_1 a_{n-2}) \circ \cdots \circ (a_1 a_2)$

b) Dokazati da se transpozicijama oblika $(1\,k)$ može generisati svaka permutacija iz S_n.

a) $\pi(a_1) = a_2$
$\pi(a_2) = a_3$
$\pi(a_3) = a_4$
$\vdots$
$\pi(a_n) = a_1$

Primjer:
$(1\ 4\ 3\ 2\ 5) =$
$= (1\ 5) \circ (1\ 2) \circ (1\ 3) \circ (1\ 4)$

Ciklus dužine n se može predstaviti kao proizvod $n-1$ transpozicija.

$sgn(\pi) = (-1)^{n-1}$

b) $\pi \in S_n \longmapsto$ proizvod ciklusa $\mapsto$ proizvod transpozicija

$$(a\ b) = (1\ a)(1\ b)(1\ a)$$

$1 \to 1$

$a \to b$

$b \to a$

Primjer:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 1 & 3 & 2 & 7 \end{pmatrix} =$$

$$= (1\ 4)(2\ 5\ 3\ 6)(7) =$$

$$= (1\ 4)(2\ 6)(2\ 3)(2\ 5) =$$

$$= (1\ 4)\underbrace{(1\ 2)(1\ 6)(1\ 2)}(1\ 2)(1\ 3)(1\ 2)\ldots(1\ 2)(1\ 5)(1\ 2)$$

8. S_n, za $n \geq 3$, nije komutativna grupa.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 1 & 2 & - & || - \end{pmatrix}$$

$$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 3 & - & || - \end{pmatrix}$$

$$\left.\begin{aligned} \pi_1 \circ \pi_2 &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 3 & 2 & - & || - \end{pmatrix} \\ \pi_2 \circ \pi_1 &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 2 & 1 & - & || - \end{pmatrix} \end{aligned}\right\} \pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1$$

$(S_3, \circ)$ - grupa

$|S_3| = 3! = 6$

9. Dokazati da je ciklus dužine n reda n.

$\pi = (a_1 a_2 \cdots a_n)$

$\underbrace{\pi \circ \cdots \circ \pi}_{k} = id$

$\pi^k = \pi^{k-1} \circ \pi$

$\pi^k(a_1) = \pi^{k-1}(a_2) = \pi^{k-2}(a_3) = \cdots = a_{(1+k) \bmod n}$

$\forall i \quad \pi^k(a_i) = a_{(i+k) \bmod n} = a_i$

$(i+k)_{\bmod n} = i \;,\; i \in \{1, \ldots, n\} \Rightarrow k = n$

(10.) Dokazati da je red permutacije NZS dužina njenih disjunktnih ciklusa.

$\pi = t_1 \circ \cdots \circ t_r$

$\pi^n = (t_1 \circ \cdots \circ t_r)^n = id_X$

$\underbrace{(t_1 \circ \cdots \circ t_r)(t_1 \circ \cdots \circ t_r) \cdots (t_1 \circ \cdots \circ t_r)}_{n} = id_X$

$t_1^n \circ t_2^n \circ \cdots \circ t_r^n = id_X$

$t_i^n = id_X$, u suprotnom $\exists a \in t_s \;\; t_s^n(a) \neq a$

$$\underbrace{t_s^n(a) = b}_{id(a) = b} \;\; ⚡$$

$t_i^{|t_i|} = id_X$

$\Downarrow$? (~~zad~~ Lema)

$ord(t_i) = |t_i| \mid n \;,\; \forall i$

$\Downarrow$

$n = nzs(|t_1|, \ldots, |t_r|)$

L/ Neka je $(G, \cdot)$ grupa i $a \in G$, $\mathrm{ord}(a) = m$.
Tada $a^n = e \iff m \mid n$.

$(\Leftarrow)$ $m \mid n \Rightarrow n = mq$

$$a^n = a^{mq} = (\underbrace{a^m}_{e})^q = e$$

$(\Rightarrow)$ $a^n = e \overset{?}{\Rightarrow} m \mid n$

Pretpostavimo da $m \nmid n$. Tada $n = mq + r$, $0 < r < m$

$$a^n = e \Rightarrow a^{mq+r} = e$$

$$(\underbrace{a^m}_{e})^q \cdot a^r = e \Rightarrow a^r = e \quad \lightning$$

(11.) a) Naći red permutacije $\alpha = (2\ 4\ 5)(1\ 8)(6\ 7\ 3\ 9)$

$$\left.\begin{array}{l} t_1 = (2\ 4\ 5) \\ t_2 = (1\ 8) \\ t_3 = (6\ 7\ 3\ 9) \end{array}\right\} \mathrm{ord}\,\alpha = \mathrm{NZS}(3, 2, 4) = 12$$

b) $\alpha = (1\ 3\ 4)(2\ 3\ 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 6 & 1 & 5 & 2 \end{pmatrix} =$

$= (1\ 3\ 6\ 2\ 4)^{(5)} \Rightarrow \mathrm{ord}(\alpha) = 5$

c) $\alpha = (1\ 2\ 3)(4\ 3\ 5)(1\ 3\ 4\ 6) = \cdots =$

$= (1\ 5\ 4\ 6\ 2\ 3) \Rightarrow \mathrm{ord}(\alpha) = 6$

Vježbe

Lagranžova teorema

Neka je G konačna grupa. Tada red svake njene podgrupe dijeli red grupe.

1. Neka su $H_1, H_2 \leq G$ (G je konačna grupa)
Ako su redovi podgrupa H_1 i H_2 uzajamno prosti brojevi, tada je $H_1 \cap H_2 = \{e\}$.

$H_1 \cap H_2 \leq H_1$ $\qquad |H_1| = n$

$H_1 \cap H_2 \leq H_2$ $\qquad |H_2| = m$

$|H_1 \cap H_2| = k$

L.T. $\Rightarrow k | n$ i $k | m \Rightarrow k = 1$ jer je $NZD(m,n)=1$

$\Rightarrow |H_1 \cap H_2| = 1$

$H_1 \cap H_2 = \{e\}$

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